Extraction of a squeezed state in a field mode via repeated measurements on an auxiliary quantum particle

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The dynamics of a system, consisting of a particle initially in a Gaussian state interacting with a field mode, under the action of repeated measurements performed on the particle, is examined. It is shown that regardless of its initial state the field is distilled into a squeezed state. The dependence on the physical parameters of the dynamics is investigated.

I. INTRODUCTION

Obtaining pure states through distillation processes is a key step to initialize and to control quantum systems in the field of quantum technology [1, 2]. For instance, state preparation is fundamental in several algorithms in quantum information and computation. So far various protocols for distillation have been proposed [2, 3, 4, 5].

A procedure for distillation through Zeno-like measurements has been presented in [6, 7]. The protocol there considered regards a generic bipartite system made of two interacting subsystems P and F. The unitary dynamics of the total system is interrupted at regular intervals by measurements performed on one of the two subsystems, say P. These measurements affect strongly the dynamics of the non-measured system F, which is then governed by some effective evolution operator. It is shown that if the spectrum of this operator satisfies certain conditions, then the non-measured system itself is driven toward a pure state irrespective of its initial conditions. This final state depends on the parameters of the total Hamiltonian, the measurement one performs, i.e., the state on which the system is projected by the measurement, and the time interval between measurements. In this distillation protocol measurements are performed repeatedly as in the case of quantum Zeno effect [8, 9, 10, 11, 12] but here the interval between measurements is kept small but finite considering the limit of large number of measurements.

This general procedure can be useful to initialize quantum systems and it has been also extended to the case where the system is made of many parts showing that it is possible to distillate entangled states [7, 13]. For example it has been shown that it is possible to establish entanglement between two separate systems via repeated measurements on a third entanglement mediator [14]. The effect of environmental noise on the protocol has been also investigated, showing that in the case of only dephasing effects one can still extract pure states [15].

However, these protocols have been analyzed only when the measured system has a discrete spectrum. Be-

cause of the conditions required for distillation, it is not obvious that distillation can be obtained even when the measured system has a continuous spectrum. The main purpose of this paper is thus to apply the general procedure, as considered in [6], to the case where the measured system has a continuous spectrum. The system considered is a particle (system P), characterized by a continuous spectrum, interacting with a field mode (system F). Our protocol consists of measurements performed on the particle P to confirm it to be in a Gaussian state. We will show that the field mode is driven toward a pure state and we will examine how this distillation process depends on parameters like the mode frequency ω, the frequency of measurements τ^{-1}, the coupling between the particle and the field mode g, and the characteristics of the Gaussian state of the particle P to be confirmed repeatedly.

The paper is organized as follows. In Sec. II we describe the features of the model we consider. In Sec. III we describe the protocol and in Sec. IV we obtain and diagonalize the projected evolution operator. In Sec. V we discuss distillation in terms of physical parameters of the system. In Sec. VI we summarize and discuss our results. In Appendices A and B we collect some of the calculations, which are omitted in the text in order to keep the readability.

II. MODEL

We consider a particle of mass m interacting with a single field mode of frequency ω. The particle interacts linearly with the field mode and the Hamiltonian reads as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + g\hat{p}(\hat{a}^\dagger + \hat{a}), \qquad (1)$$

where $\hat{p}$ is the particle momentum operator, $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the field mode satisfying the commutation rules, $[\hat{a}, \hat{a}^\dagger] = 1$, etc., and the real parameter g is the coupling constant. This Hamiltonian is analogous to the electromagnetic Hamil-

tonian restricted to the single-mode case and in dipole approximation [16, 17].

In the interaction picture the interaction Hamiltonian of Eq. (1) at time s takes the form

$$\hat{H}^I(s) = g\hat{p}(\hat{a}^\dagger e^{i\omega s} + \hat{a} e^{-i\omega s}). \qquad (2)$$

The evolution determined by this Hamiltonian can be treated exactly since its commutator at two different times,

$$[\hat{H}^I(s'), \hat{H}^I(s'')] = -2ig^2\hat{p}^2 \sin\omega(s' - s''), \qquad (3)$$

commutes with the Hamiltonian itself. This allows one to obtain the exact evolution operator [16, 18] as

$$\hat{U}^I(\tau) = T_\leftarrow \exp\left[-\frac{i}{\hbar}\int_0^\tau ds\, \hat{H}^I(s)\right]$$

$$= \exp\left[-\frac{1}{2\hbar^2}\int_0^\tau ds \int_0^\tau ds'\, \theta(s - s')[\hat{H}^I(s), \hat{H}^I(s')]\right]$$

$$\times \exp\left[-\frac{i}{\hbar}\int_0^\tau ds\, \hat{H}^I(s)\right], \qquad (4)$$

where $T_\leftarrow$ is the time ordering operator and $\theta(s - s')$ is the Heaviside step function. Using Eqs. (2)–(3) and

$$\int_0^\tau ds \int_0^\tau ds'\, \theta(s - s') \sin\omega(s - s') = \frac{\omega\tau - \sin\omega\tau}{\omega^2}, \qquad (5)$$

the time evolution operator at time τ can be put in the form

$$\hat{U}^I(\tau) = \exp\left[ig^2\hat{p}^2 \frac{\omega\tau - \sin\omega\tau}{\hbar^2\omega^2}\right]$$

$$\times \exp\left[g\hat{p}\left(\hat{a}^\dagger \frac{1 - e^{i\omega\tau}}{\hbar\omega} - \hat{a}\frac{1 - e^{-i\omega\tau}}{\hbar\omega}\right)\right]. \qquad (6)$$

Indicating with $\hat{U}_0$ the time evolution operator associated to the free part of the Hamiltonian of Eq. (1), the time evolution operator in the Schrödinger picture, $\hat{U}(\tau) = \hat{U}_0(\tau)\hat{U}^I(\tau)$, is given by

$$\hat{U}(\tau) = \exp\left[-\frac{i}{\hbar}\frac{\hat{p}^2\tau}{2m}\left(1 - \frac{2mg^2}{\hbar\omega}\frac{\omega\tau - \sin\omega\tau}{\omega\tau}\right)\right]$$

$$\times \exp\left[-i\omega\tau\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right]$$

$$\times \exp\left[g\hat{p}\left(\hat{a}^\dagger \frac{1 - e^{i\omega\tau}}{\hbar\omega} - \hat{a}\frac{1 - e^{-i\omega\tau}}{\hbar\omega}\right)\right]. \qquad (7)$$

III. PROTOCOL

In this section we briefly review the distillation procedure based on repeated measurements [6] as applied to our system.

Assume that at time $t = 0$ the particle is prepared in a pure state $|\Phi_0\rangle$ and the field in an arbitrary mixed state,

denoted by $\rho_F(0)$. The unitary dynamics of the total system, governed by the time-evolution operator $\hat{U}(\tau)$ in Eq. (7), is interrupted by the measurements performed on the particle at intervals τ. Each time, the measurement projects the particle in its initial state $|\Phi_0\rangle$. This action is represented by the projection operator $\mathcal{O} = |\Phi_0\rangle\langle\Phi_0| \otimes \mathbf{1}_F$. The projection is partial on the total system because only the state of the particle is set back to its initial state while the field is not initialized, even though its dynamics is certainly affected by the measurements. The total system after N measurements is described by

$$\hat{\rho}_T^\tau(N) \propto [\mathcal{O}\hat{U}(\tau)]^N[|\Phi_0\rangle\langle\Phi_0| \otimes \hat{\rho}_F(0)][\hat{U}^\dagger(\tau)\mathcal{O}]^N. \qquad (8)$$

Following [6], we introduce the projected evolution operator between two consecutive measurements,

$$\hat{V}_\tau = \langle\Phi_0|\hat{U}(\tau)|\Phi_0\rangle, \qquad (9)$$

so that, after N measurements on the particle, the field is described by the density matrix

$$\hat{\rho}_F^\tau(N) = \frac{\hat{V}_\tau^N \hat{\rho}_F(0) \hat{V}_\tau^{\dagger N}}{P_\tau(N)}. \qquad (10)$$

Note that we retain only events in which the particle P is found in the state $|\Phi_0\rangle$ by every measurement. The normalization factor

$$P_\tau(N) = \text{Tr}_F\{\hat{V}_\tau^N \hat{\rho}_F(0) \hat{V}_\tau^{\dagger N}\} \qquad (11)$$

represents such a probability (up to N measurements) and gives the probability to obtain the state (10).

In this paper, we consider the following protocol: the particle P is initially prepared in a Gaussian state $|\Phi_0\rangle$ and repeatedly projected on it at intervals τ. Such a measurement would be realized by switching on a harmonic potential, whose ground state coincides with the Gaussian state $|\Phi_0\rangle$, and seeing whether P is in the ground state. The Gaussian state $|\Phi_0\rangle$ is characterized by the variances of the coordinate Δr_0 and of the momentum Δp_0, satisfying $\Delta r_0 \Delta p_0 = \hbar/2$. In the momentum space, it is given by

$$|\Phi_0\rangle = \int dp\, \frac{1}{\sqrt[4]{2\pi(\Delta p_0)^2}} \exp\left[-\frac{p^2}{4(\Delta p_0)^2}\right]|p\rangle, \qquad (12)$$

where $|p\rangle$ are the eigenstates of the momentum operator $\hat{p}$, the initial average momentum is $p_0 = 0$, and the initial average position $r_0 = 0$.

Once chosen the measurements to perform on the particle, the next step is to ask if the protocol considered leads to purification of the field mode. In order to achieve purification for the non-measured system, the spectrum of the projected evolution operator $\hat{V}_\tau$ defined in Eq. (9) must satisfy the following conditions [6]: its largest (in magnitude) eigenvalue must be unique, discrete, and non-degenerate. In the next section we shall compute the projected evolution operator $\hat{V}_\tau$ for the present setup and then diagonalize it in order to analyze its spectrum.

IV. PROJECTED EVOLUTION OPERATOR

Both the field evolution and the survival probability, given by Eqs. (10) and (11), depend essentially on the projected evolution operator $\hat{V}_\tau$ defined in Eq. (9). This operator, using Eq. (12) as the state of the particle $|\Phi_0\rangle$ to be measured repeatedly, results in

$$\hat{V}_\tau = \exp\left[-i\omega\tau\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right]\frac{1}{\sqrt{2\pi(\Delta p_0)^2}}\int dp\, e^{-f(p)},\tag{13}$$

where

$$f(p) = \beta\frac{p^2}{2(\Delta p_0)^2}$$
$$- ig\frac{\sqrt{2(1-\cos\omega\tau)}}{\hbar\omega}(\hat{a}^\dagger e^{i\frac{\omega\tau}{2}} + \hat{a}\,e^{-i\frac{\omega\tau}{2}})p,\tag{14a}$$
$$\beta = 1 + i\frac{(\Delta p_0)^2\tau}{\hbar m}\left(1 - \frac{2mg^2}{\hbar\omega}\frac{\omega\tau - \sin\omega\tau}{\omega\tau}\right).\tag{14b}$$

Introducing three independent dimensionless parameters

$$\bar{\tau} = \omega\tau, \qquad \bar{g} = g\sqrt{\frac{m}{\hbar\omega}}, \qquad \bar{\Delta}_p = \frac{\Delta p_0}{\sqrt{m\hbar\omega}},\tag{15}$$

and performing the integration in Eq. (13) we obtain the projected evolution operator as

$$\hat{V}_\tau = M\exp\left[-i\omega\tau\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right]$$
$$\times \exp\left[-\frac{G}{2}(\hat{a}^\dagger e^{i\frac{\omega\tau}{2}} + \hat{a}\,e^{-i\frac{\omega\tau}{2}})^2\right],\tag{16}$$

where

$$M = \frac{1}{\sqrt{1 + i\bar{\Delta}_p^2\bar{\tau}\left[1 - 2\bar{g}^2\left(1 - \frac{\sin\bar{\tau}}{\bar{\tau}}\right)\right]}},\tag{17a}$$

$$G = 2M^2\bar{g}^2\bar{\Delta}_p^2(1 - \cos\bar{\tau}).\tag{17b}$$

A. Diagonalization

As already stated, in order to check if the conditions for purification are satisfied in the present setup we need to analyze the spectrum of $\hat{V}_\tau$. To diagonalize the projected evolution operator $\hat{V}_\tau$ it is useful to rewrite its expression in Eq. (16) in a single unified exponential. The details are given in Appendix A, where the projected evolution operator $\hat{V}_\tau$ is arranged in Eq. (A10) in the form

$$\hat{V}_\tau = M\exp\left\{\frac{\ln(q - \sqrt{q^2-1})}{\sqrt{q^2-1}}G\right.$$
$$\left.\times\left[\frac{\hat{a}^{\dagger 2} + \hat{a}^2}{2} + \tilde{q}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right]\right\},\tag{18}$$

where

$$q = \cos\bar{\tau} + iG\sin\bar{\tau}, \qquad \tilde{q} = \cos\bar{\tau} + \frac{i}{G}\sin\bar{\tau}.\tag{19}$$

It is diagonalized by the similarity transformation

$$e^{\eta\hat{A}}e^{\zeta\hat{A}^\dagger}\hat{V}_\tau e^{-\zeta\hat{A}^\dagger}e^{-\eta\hat{A}},\tag{20}$$

where

$$\zeta = \tilde{q} \pm \sqrt{\tilde{q}^2 - 1}, \qquad \eta = \pm\frac{1}{2\tilde{q}},\tag{21}$$

and then $\hat{V}_\tau$ is transformed to

$$\hat{V}_\tau \to M\exp\left[\mp\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\ln(q - \sqrt{q^2-1})\right].\tag{22}$$

This shows that the eigenvalues γ_n and the right and left eigenstates, $|u_n\rangle$ and $\langle v_n|$, of

$$\hat{V}_\tau = \sum_n \gamma_n|u_n\rangle\langle v_n|\tag{23}$$

are given by

$$\gamma_n = M\exp\left[\mp(n + 1/2)\ln(q - \sqrt{q^2-1})\right],\tag{24a}$$
$$|u_n\rangle = e^{-\zeta\hat{A}^\dagger}e^{-\eta\hat{A}}|n\rangle, \quad \langle v_n| = \langle n|e^{\eta\hat{A}}e^{\zeta\hat{A}^\dagger},\tag{24b}$$

where $|n\rangle$ is the eigenstate of the number operator $\hat{a}^\dagger\hat{a}$ belonging to its eigenvalue $n = 0, 1, \ldots$

A comment is in order as to the choice of the sign $(\pm)$ in Eqs. (21), (22), and (24). As shown in Appendix B, in order to assure the normalizability of the eigenstates, for a given q, a correct sign must be chosen so that the inequality $|q \pm \sqrt{q^2-1}| = |q \mp \sqrt{q^2-1}|^{-1} < 1$ holds. The real part of $\ln(q - \sqrt{q^2-1})$ is equal to $\ln|q - \sqrt{q^2-1}| = -\ln|q + \sqrt{q^2-1}|$ and it is easy to verify that one has to choose the upper(lower) sign, that is, $+(-)$ in Eq. (21) and $-(+)$ in Eqs. (22) and (24) if the real part is positive(negative). Therefore, $\pm$ may be substituted with the sign of the real part of $\ln(q - \sqrt{q^2-1})$. We observe that when $\bar{\tau} = (2\ell + 1)\pi$ $(\ell = 0, 1, \ldots)$ we have $q = 0$ and then $|q \pm \sqrt{q^2-1}| = 1$, independently of the choice of sign, so that for these values of $\bar{\tau}$ the normalizability of the eigenstates is lost and there is no distillation.

V. DISTILLATION

Analyzing the structure of the eigenvalues γ_n of $\hat{V}_\tau$ in Eq. (24) we see that the largest (in magnitude) eigenvalue γ_0 is unique, discrete, and nondegenerate, so that in the large N limit, the operator $\hat{V}_\tau^N$ is dominated by a single term [6]

$$\hat{V}_\tau^N \xrightarrow{\text{large } N} \gamma_0^N|u_0\rangle\langle v_0|.\tag{25}$$

Then, using Eq. (25) in Eq. (10), in the large N limit for a nonvanishing τ, the state of the field asymptotically approaches the pure state,

$$\hat{\rho}_F^\tau(N) \xrightarrow{\text{large } N} \frac{|u_0\rangle\langle u_0|}{\langle u_0|u_0\rangle} = |\xi\rangle\langle\xi|. \qquad (26)$$

Notice that the pure state $|u_0\rangle$ is explicitly written as

$$|u_0\rangle = \mathrm{e}^{-\zeta \hat{A}^\dagger}|0\rangle = \sqrt{\cosh r}\,\hat{S}(\xi)|0\rangle = \sqrt{\cosh r}|\xi\rangle, \quad (27)$$

where $\hat{S}(\xi) = \exp(-\frac{\xi}{2}\hat{a}^{\dagger 2} - \frac{\xi^*}{2}\hat{a}^2)$ is a squeezing operator with $\xi = r\mathrm{e}^{i\varphi}$. The squeezing parameter r and the phase φ are given by

$$r = \tanh^{-1}|\zeta|, \qquad \varphi = \arg\zeta, \qquad (28)$$

where ζ is defined in Eq. (21) and $\tilde{q}$ in Eq. (19). Therefore, Eq. (26) shows that the field mode is distilled into a squeezed state. The final pure squeezed state $|\xi\rangle$ is independent of the choice of the initial state of the field, i.e., any initial (eventually mixed) state shall be driven to the unique pure state $|\xi\rangle$ by the repeated measurements performed on the particle to confirm it to be in the Gaussian state $|\Phi_0\rangle$.

There are four, the first two independent of and the last two dependent on the initial state of the field mode, relevant quantities characterizing this distillation process:

- Speed of distillation: the purification is achieved quickly if $|\gamma_n/\gamma_0| = |\gamma_1/\gamma_0|^n \ll 1$ for $n \neq 0$. This means that we have a quick distillation if

$$|\gamma_1/\gamma_0| = \exp\left[-\left|\ln\left|q - \sqrt{q^2-1}\right|\right|\right] \ll 1. \qquad (29)$$

  The quickness of distillation is thus linked to $\left|\ln\left|q - \sqrt{q^2-1}\right|\right| = \left|\ln\left|q + \sqrt{q^2-1}\right|\right|$.

- Degree of squeezing of the distilled state $|\xi\rangle$: this is given by r in Eq. (28). It regulates the average number of quanta $\langle \hat{a}^\dagger \hat{a}\rangle = \sinh^2 r$ and its variance $\langle(\hat{a}^\dagger\hat{a})^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2 = 2\sinh^2 r\,(\sinh^2 r + 1)$. For a given r, φ determines the variance $\Delta_{\hat{x}_\vartheta}^2 = \langle(\hat{x}_\vartheta)^2\rangle - \langle\hat{x}_\vartheta\rangle^2$ of the quadrature operator $\hat{x}_\vartheta = \frac{1}{\sqrt{2}}(\hat{a}\,\mathrm{e}^{-i\vartheta} + \hat{a}^\dagger\mathrm{e}^{i\vartheta})$, where ϑ is a real phase. Observe that $\mathrm{e}^{-2r}/2 < \Delta_{\hat{x}_\vartheta}^2 < \mathrm{e}^{2r}/2$. If $\sin^2(\vartheta - \varphi) < (\mathrm{e}^{2r} + 1)^{-1}$ the variance $\Delta_{\hat{x}_\vartheta}^2$ is less than $1/2$ and in this case the quadrature $\hat{x}_\vartheta$ is squeezed.

- Probability of success of the protocol: this is represented by the survival probability $P_\tau(N)$ introduced in Eq. (11). It is desirable to have a higher probability of success to keep higher yields when the distillation has been attained. While the state approaches the squeezed state $|\xi\rangle$ as shown in (26), the probability behaves as

$$P_\tau(N) \xrightarrow{\text{large } N} |\gamma_0|^{2N}\langle u_0|u_0\rangle\langle v_0|\hat{\rho}_F(0)|v_0\rangle. \qquad (30)$$

  Therefore, it is preferable to have a larger $|\gamma_0|$ (closer to unity) for a slower decay of the probability $P_\tau(N)$.

- Fidelity: it indicates how close the extracted state of the field mode is to the target pure state after N measurements. For an efficient distillation protocol the state after N measurements must be as close to the final target state as possible. The fidelity is defined by

$$F_\tau(N) = \langle\xi|\hat{\rho}_F^\tau(N)|\xi\rangle, \qquad (31)$$

  which is a positive number and approaches unity as the field state $\hat{\rho}_F^\tau(N)$ becomes close to the target pure state $|\xi\rangle$.

In the following we shall analyze the dependence of these quantities on the parameters $\bar{\tau}$, $\bar{g}$, and $\bar{\Delta}_p$. Our aim is to find optimal values of the parameters satisfying two independent requirements:

(i) fast distillation with high degree of squeezing [see (29)], and

(ii) high fidelity with a sufficiently high probability of success of the protocol [see (30)].

Even if the second condition is fundamental for an efficient protocol for the distillation, we start discussing the first condition, concerning the speed of distillation and the degree of squeezing, because it involves quantities which are independent of the initial state of the field mode.

A. Distillation speed vs squeezing

In order to have indications for the ranges of the parameters where there is a quick distillation, we plot the ratio between the first two eigenvalues $-\ln|\gamma_1/\gamma_0| = \left|\ln\left|q \mp \sqrt{q^2-1}\right|\right|$, looking for regions where this quantity becomes large. Figure 1 shows $-\ln|\gamma_1/\gamma_0|$ as a function of the dimensionless parameters in Eq. (15), $\bar{\tau}$ and $\bar{g}$ with fixed $\bar{\Delta}_p = 1$. The plot evidences that, for a fixed $\bar{\Delta}_p$, the ratio $-\ln|\gamma_1/\gamma_0|$ has a strong dependence on the values of $\bar{g}$ and $\bar{\omega}$. In particular, for $\bar{\tau} \lesssim \pi$ and $\bar{g} \gtrsim 1/\sqrt{2}$, the ratio can be greater than unity and in this region a fast distillation is available.

Next, to investigate the degree of squeezing of the distilled state, we plot the hyperbolic tangent of the squeezing parameter, $\tanh r = |\zeta|$, instead of the squeezing parameter r itself, in order to avoid divergences in r when $|\zeta| \to 1$. Figure 2 shows $\tanh r$ as a function of the dimensionless parameters $\bar{\tau}$ and $\bar{g}$ with fixed $\bar{\Delta}_p = 1$. It is evident that for odd multiples of π (values of τ not allowed for distillation) and $\bar{g} \neq 0$ we have $\tanh r \to 1$ and thus $r \to \infty$. With $\bar{\tau}$ just smaller than these values the distilled state is highly squeezed and in particular the region $\bar{\tau} \lesssim \pi$ and $\bar{g} \gtrsim 1/\sqrt{2}$ appears to be more appropriate to obtain states with a high degree of squeezing. For $\bar{g}$ and $\bar{\Delta}_p$ large enough, $\tanh r$ depends only on τ.

The above plots indicate that distillation speed, linked to $-\ln|\gamma_1/\gamma_0|$, and the degree of squeezing, linked to
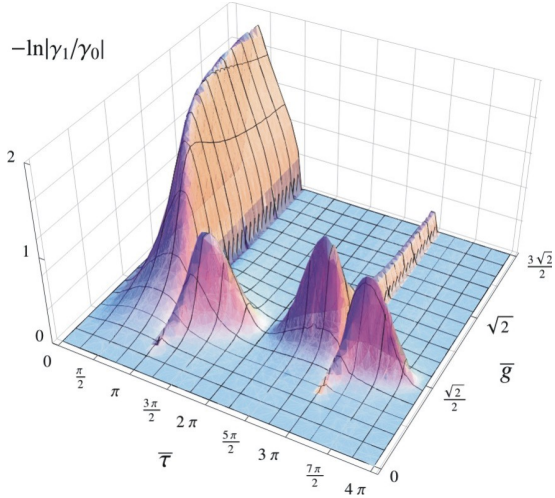
FIG. 1: The ratio between the first two eigenvalues $-\ln|\gamma_1/\gamma_0|$ as a function of the dimensionless parameters $\bar{\tau}$ and $\bar{g}$ with fixed $\bar{\Delta}_p = 1$.
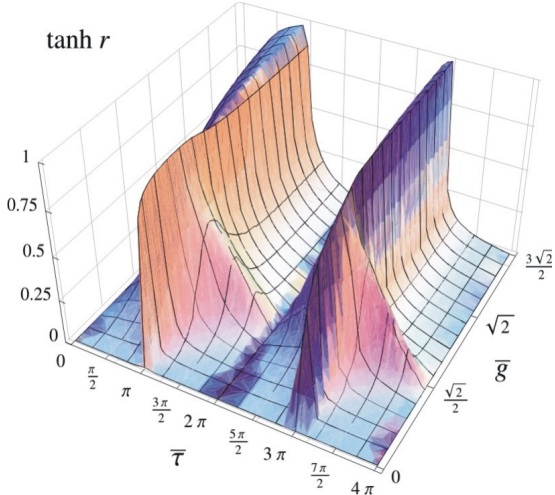

FIG. 2: $\tanh r$ as a function of the dimensionless parameters $\bar{\tau}$ and $\bar{g}$ with fixed $\bar{\Delta}_p = 1$.

$\tanh r$, have strong and different dependencies on the parameters. Now we show that it is possible to find values of the parameters where quick distillation and strong squeezing would be attainable simultaneously. In Fig. 3 we compare the behaviors of $-\ln|\gamma_1/\gamma_0|$ (dotdashed line) and of $\tanh r$ (solid line), as functions of the dimensionless parameter $\bar{\tau}$ with $\bar{g} = 1$ and $\bar{\Delta}_p = 0.4$. From the plot one sees that $-\ln|\gamma_1/\gamma_0| \gtrsim 0.5$ (sufficiently quick distillation) and $\tanh r \gtrsim 0.5$ (sufficiently strong squeezing) can be fulfilled for $\pi/2 \lesssim \omega\tau \lesssim \pi$.

B. Survival probability vs fidelity

Both the survival probability and the fidelity depend on the initial state of the field. We consider the case where the field is initially in a coherent state $|\alpha\rangle$. Al-
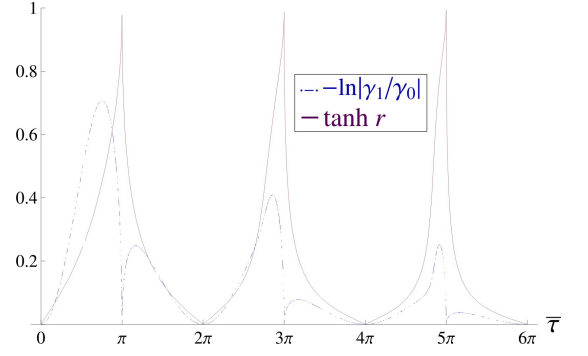

FIG. 3: $-\ln|\gamma_1/\gamma_0|$ (dotdashed line) and $\tanh r$ (solid line), as functions of the dimensionless parameter $\bar{\tau}$ with $\bar{g} = 1$ and $\bar{\Delta}_p = 0.4$.
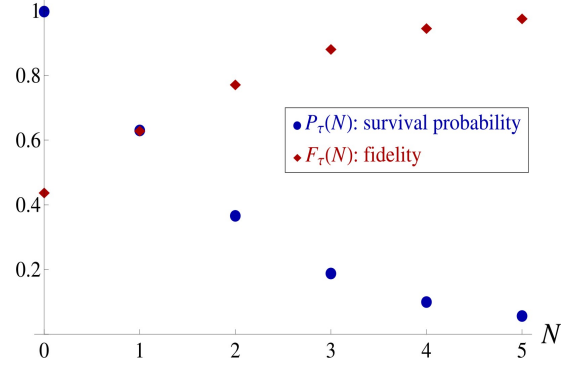

FIG. 4: $P_\tau(N)$ (circles) vs $F_\tau(N)$ (diamonds) as functions of the number of measurements N with $\bar{g} = 1$, $\bar{\Delta}_p = 0.4$, and $\bar{\tau} = 0.9\pi$. The field is initially in a coherent state $|\alpha\rangle$ of amplitude $\alpha = 1$.

though the survival probability and the fidelity can be analytically obtained when the field is initially in a coherent state, their explicit expressions are rather involved so that in the following we will just present the results.

Now, in order to check if the present protocol gives distillation with a good probability of success in this case, we compare the evolutions of the survival probability and of the fidelity for a given τ. In particular we are looking for values of the parameters such that when the fidelity gets close to 1 the survival probability is still high enough. In Fig. 4 the evolutions of the survival probability and of the fidelity for $\alpha = 1$ and $\bar{\tau} = 0.9\pi$ are given. The values of the other parameters are the same as in Fig. 3. From the plot one sees that high values of fidelity can be obtained with the survival probability still far from 0.

These two plots, Figs. 3 and 4, clearly indicate that our protocol allows one to generate efficiently, that is, with a high fidelity and a finite (nonvanishing) probability, pure squeezed states with a sufficiently high degree of squeezing. Indeed, for example, if we tune the parameters as $\bar{g} = 1$, $\bar{\Delta}_p = 0.4$ and perform measurements with a period of $\bar{\tau} = 0.9\pi$, we will obtain a well squeezed state with a squeezing parameter $r \sim \tanh^{-1} 0.6 \sim 0.54$ with a fidelity of $\sim 80\%$ and a probability $\sim 20\%$ already after

a couple of measurements $N \sim 2, 3$.

VI. CONCLUSIONS

In general for a bipartite system made of two interacting parts it is known that Zeno-like measurements on a part may lead, under certain conditions, the non-measured part towards a pure state independently of its initial configuration. The general procedure has been so far analyzed in the case where the measured part has a discrete spectrum, while it is not obvious whether the distillation can be obtained when the measured system has a continuous spectrum. Here we have analyzed this topic considering a specific bipartite system consisting of a particle, characterized by a continuous spectrum, interacting with a field mode characterized by a discrete spectrum but with an infinite number of levels. The present distillation protocol consists of repeatedly projecting the particle to a Gaussian state. The projected evolution operator that regulates the field-mode dynamics between the consecutive measurements performed on the particle has been obtained. It has been shown that, with the measurement protocol chosen, the spectrum of this operator is discrete and satisfies the criteria that allow one to obtain a field-mode distillation. As a consequence of the protocol, the field is driven to a squeezed state independently of its initial state. The dependencies of the distillation speed, that is connected to the ratio between the first two eigenvalues of the projected evolution operator, and of the characteristics of the distilled state, i.e., the squeezing parameter, are investigated as functions of parameters such as the interval between two measurements, the particle-field mode coupling constant, and the width of the particle's Gaussian state. Varying the values of the parameters different regimes are observed and we have shown that it is possible to choose values such that one has both quick distillation and strong squeezing and/or high values of fidelity and finite, well far from zero, values of probability of success of the measurement protocol.

APPENDIX A: UNIFICATION OF EXPONENTIAL FACTORS IN V_τ

The projected time-evolution operator $\hat{V}_\tau$ of Eq. (16) is of the following form:

$$\hat{V}_\tau = M e^{-i\omega\tau \hat{B}} e^{-G(e^{i\omega\tau}\hat{A}^\dagger + e^{-i\omega\tau}\hat{A} + \hat{B})}, \qquad \text{(A1)}$$

where

$$\hat{A} = \frac{1}{2}\hat{a}^2, \qquad \hat{B} = \hat{a}^\dagger\hat{a} + \frac{1}{2}. \qquad \text{(A2)}$$

Note the Lee algebra among $\hat{A}$, $\hat{A}^\dagger$, and $\hat{B}$,

$$[\hat{A}, \hat{A}^\dagger] = \hat{B}, \qquad [\hat{A}, \hat{B}] = 2\hat{A}, \qquad [\hat{A}^\dagger, \hat{B}] = -2\hat{A}^\dagger. \qquad \text{(A3)}$$

For these generators of the algebra, the following formula for the factorization of exponential is available:

$$e^{\mu\hat{A}^\dagger + \nu\hat{A} + \lambda\hat{B}} = e^{x\hat{A}^\dagger} e^{y\hat{B}} e^{z\hat{A}}, \qquad \text{(A4)}$$

where

$$\begin{cases} x = \dfrac{(\mu/\kappa)\tanh\kappa}{1 - (\lambda/\kappa)\tanh\kappa}, \\[2mm] y = -\dfrac{1}{2}\ln\left(\cosh\kappa - \lambda\dfrac{\sinh\kappa}{\kappa}\right)^2, \quad \kappa = \sqrt{\lambda^2 - \mu\nu}. \\[2mm] z = \dfrac{(\nu/\kappa)\tanh\kappa}{1 - (\lambda/\kappa)\tanh\kappa}, \end{cases} \qquad \text{(A5)}$$

Inverse relations are given by

$$\begin{cases} \mu = \dfrac{\kappa}{\sinh\kappa} x e^{-y}, \\[2mm] \nu = \dfrac{\kappa}{\sinh\kappa} z e^{-y}, \quad \kappa = \cosh^{-1}\dfrac{1}{2}(e^y + e^{-y} - xze^{-y}). \\[2mm] \lambda = -\sqrt{\kappa^2 + \mu\nu}, \end{cases} \qquad \text{(A6)}$$

Note the formula for the reciprocal function

$$\cosh^{-1}x = \pm\ln(x + \sqrt{x^2 - 1}). \qquad \text{(A7)}$$

Now, let us come back to the projected time-evolution operator (A1). By making use of the factorization formula (A4)–(A5), one has

$$\hat{V}_\tau = M e^{-i\omega\tau\hat{B}} e^{-\frac{G}{1+G}e^{i\omega\tau}\hat{A}^\dagger} e^{-\ln(1+G)\hat{B}} e^{-\frac{G}{1+G}e^{-i\omega\tau}\hat{A}}. \qquad \text{(A8)}$$

By exchanging the order of the first two exponentials, one obtains

$$\hat{V}_\tau = M e^{-x\hat{A}^\dagger} e^{-y\hat{B}} e^{-x\hat{A}}, \qquad \begin{cases} x = \dfrac{G}{1+G}e^{-i\omega\tau}, \\[2mm] y = \ln(1+G) + i\omega\tau. \end{cases} \qquad \text{(A9)}$$

Then, we unify the exponentials via the formula (A4) with (A6) to obtain

$$\hat{V}_\tau = M e^{\mu(\hat{A}^\dagger + \hat{A}) + \lambda\hat{B}},$$

$$\begin{cases} \mu = \dfrac{G}{\sqrt{q^2-1}} \ln(q - \sqrt{q^2-1}) \\[2mm] \lambda = \dfrac{G\cos\omega\tau + i\sin\omega\tau}{\sqrt{q^2-1}} \ln(q - \sqrt{q^2-1}), \end{cases} \qquad \text{(A10)}$$

where q is defined in (19). The expression (18) is thus obtained.

APPENDIX B: CHOICE OF SIGN IN DIAGONALIZATION

Equations (21), (22), and (24) seem to show that the diagonalization procedure presented in the text admits two possible signs $\pm$. The immediate concern, in such a case, would be whether the unitarity of the time-evolution operator $\hat{U}(\tau)$, which dictates that the absolute values of eigenvalues $\hat{V}_\tau$ are strictly upper-bounded by unity, is preserved by the solution in Eq. (24), or which would be the right choice of the signs ($\mp$) of the eigenvalues (and the eigenstates) if only one of them can be allowed. It will be shown here that the normalizable eigenstates are those where $|q - \sqrt{q^2-1}| < 1$, i.e., belonging to the eigenvalues with their magnitudes always less than unity,

$$|\gamma_n| \le \exp\left[(n+1/2)\ln\left|q \pm \sqrt{q^2-1}\right|\right] < 1. \qquad \text{(B1)}$$

The normalizability of the eigenstate $|u_0\rangle$ for $n=0$,

$$\left\| e^{-\zeta\hat{A}^\dagger} e^{-\eta\hat{A}} |0\rangle \right\| < \infty, \qquad \text{(B2)}$$

is sufficient to show the above statement. The left-hand side is calculated to be

$$\langle 0| e^{-\zeta^*\hat{A}} e^{-\zeta\hat{A}^\dagger} |0\rangle = \int \frac{\mathrm{d}^2\alpha}{\pi} \, e^{-\frac{\zeta^*}{2}\alpha^2} e^{-\frac{\zeta}{2}(\alpha^*)^2} |\langle 0|\alpha\rangle|^2$$

$$= \int \frac{\mathrm{d}\alpha_R \, \mathrm{d}\alpha_I}{\pi} \, e^{-\alpha\cdot\mathcal{A}\alpha}, \qquad \text{(B3)}$$

where the exponent is explicitly written as

$$\alpha\cdot\mathcal{A}\alpha = \begin{pmatrix} \alpha_R & \alpha_I \end{pmatrix} \begin{pmatrix} 1+\zeta_R & \zeta_I \\ \zeta_I & 1-\zeta_R \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_I \end{pmatrix}, \qquad \text{(B4)}$$

and $\alpha_R = \mathrm{Re}\,\alpha$, $\alpha_I = \mathrm{Im}\,\alpha$, $\zeta_R = \mathrm{Re}\,\zeta$, $\zeta_I = \mathrm{Im}\,\zeta$. The eigenvalues of the matrix $\mathcal{A}$ are easily found to be $1\pm|\zeta|$, both of which have to be positive, i.e., $|\zeta| < 1$, in order for the above state is normalizable. That is, the normalizability of the eigenstates is assured if the condition $|\zeta| < 1$ is satisfied. This condition is explicitly written as that for q and G

$$|\zeta| = \left|\tilde{q} \pm \sqrt{\tilde{q}^2-1}\right| < 1, \qquad \text{(B5)}$$

which just reduces to

$$\left|q \pm \sqrt{q^2-1}\right| < 1, \qquad \text{(B6)}$$

if G is replace with G^{-1}. The normalizability condition of the eigenstates for the case of G^{-1} thus ensures the unitarity in the case of G. Stated differently, we have to choose an appropriate sign between $+$ and $-$ so that the absolute value of the argument of the logarithm satisfies the above inequality in order for the eigenstates to be normalizable. The unitarity is always satisfied, or we just have to make an appropriate choice of the phase of the square root $\sqrt{q^2-1} = \sqrt{(\cos\omega\tau + iG\sin\omega\tau)^2 - 1}$, to which the sign ($\pm$) could be considered to be absorbed.

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